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LINEAR OPERATORS AND THE EQUATIONS OF MOTION OF INFINITE LINEAR CHAINS

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OF MOTION OF INFINITE LINEAR CHAINS

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SUMMARY

The objectives of this study are (1) to reinterpret some earlier results of Law and Martens on solutions of the equations of motion of infinite linear chains in terms of the theory of linear operators and (2) to use that theory to obtain solutions for chains in which a type of symmetry previously assumed is not present. Both half-infinite and fully infinite chains are considered.

The procedure is to use the coefficients of the system of differential equations to define an operator on a dense subset of a Hilbert space. Sufficient conditions are stated for the closure of this operator to be self-adjoint. For systems where such conditions are satisfied, a solution to the equations of motion is formulated in terms of the resolution of the identity of the closure of the operator.

The problem of finding the resolution of the identity for the closure of the operator associated with the equations of motion of either a half-infinite or a fully infinite chain is discussed. Procedures for exhibiting solutions of the equations of motion are described under the assumption that integrators can be found for appropriate sequences of orthogonal polynomials whose recurrence relations are known.

Two examples of fully infinite chains are given, and solutions of their equations of motion are found by the method described. One of the chains does not possess physical symmetry about some middle mass.

Several proofs and more detailed accounts of the analytic techniques used are found in the four appendices.

CHAPTER I

INTRODUCTION

In a dissertation presented to the Division of Graduate Studies and Research at the Georgia Institute of Technology in May, 1968, A. G. Law [7] found solutions to the equations of motion of a number of half-infinite linear chains of springs and masses (see Figure 1a). In April, 1971, W. F. Martens [8] presented a dissertation to the Division of Graduate Studies and Research in which he found solutions to the equations of motion of several fully infinite linear chains of springs and masses (see Figure 1b) under the assumption that each chain possessed physical symmetry about some middle mass. The theory of linear operators played no role in either of these dissertations.

The present study has two principal goals:

(1) to reformulate the methods used by Law and Martens in terms of the theory of linear operators and

(2) by using this theory to remove the condition of physical symmetry imposed by Martens.

Chapter II is a description of the method used to achieve these goals. The first section of the chapter contains a summary of pertinent results from the theory of linear operators which are used later in the chapter. The second section is concerned with the reformulation of the solution of the equations of motion of the half-infinite chain. The coefficients of the system of differential equations are used to define

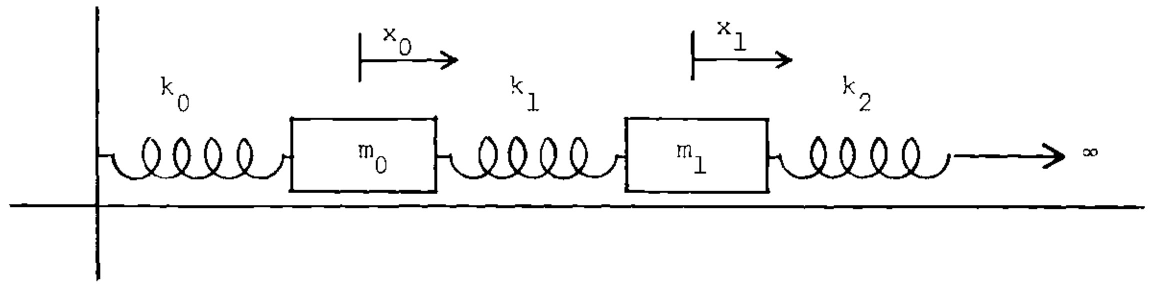

Figure 1a. A Half-Infinite Linear Chain of Springs and Masses

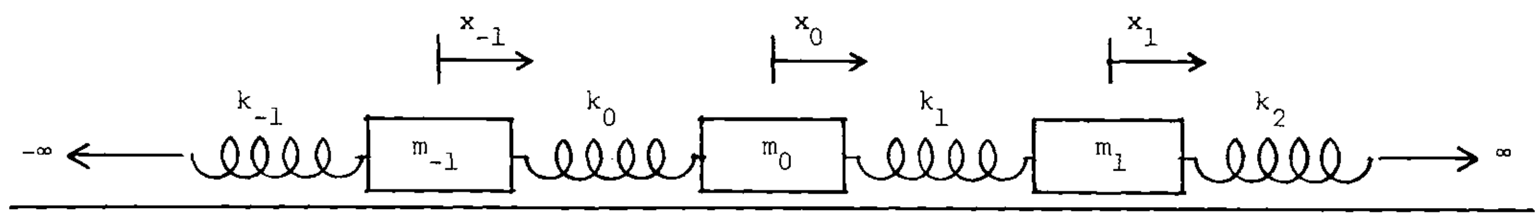

Figure 1b. A Fully Infinite Linear Chain of Springs and Masses

a linear operator on a dense subspace of the Hilbert space $\ell_2(0,\infty)$. A sufficient condition for the closure of this operator to be self-adjoint is stated, and a reference is given for other known sufficient conditions. Under the assumption that one of these sufficient conditions is satisfied, a solution is formulated in terms of the resolution of the identity of the self-adjoint operator. In the third section of the chapter a similar procedure is used to formulate a solution of the equations of motion of the fully infinite chain in terms of the resolution of the identity for a certain self-adjoint operator on $\ell_2(-\infty,\infty)$. This formulation does not require that the system have physical symmetry about some middle mass.

In Chapter III the problem of exhibiting the resolution of the identity for a known self-adjoint operator is discussed. It is shown that for the type of operator associated with the fully infinite chain, the problem of exhibiting the resolution of the identity can be reduced to the problem of finding integrators for two sequences of orthogonal polynomials whose recurrence relations are known.

In Chapter IV the procedure of Chapters II and III is used to exhibit a solution of the equations of motion of the uniform, fully infinite linear chain. The solution is in a different form from the one found by Martens [8] but is easily shown to agree with his.

In Chapter V an example of a fully infinite linear chain without physical symmetry is given. The solution of the equations of motion is found by the procedure of Chapters II and III.

Appendix A contains a proof of sufficient conditions for the self-adjointness of the closure of the types of operators associated with half-infinite and fully infinite chains. Appendix B contains a proof of the validity of differentiating under the integral sign in some integrals which appear in the solutions for the half-infinite and fully infinite chains. Appendix C describes some techniques which have proved useful in finding explicit representations of solutions of the fully infinite chain. Appendix D outlines a direct verification of the solution of the example of Chapter V.

CHAPTER II

FORMULATION OF THE PROBLEM IN TERMS OF THE THEORY OF LINEAR OPERATORS

This chapter is divided into three sections. The first is a summary of some ideas about linear operators on Hilbert spaces. In the second, these ideas are used to formulate a solution of the equations of motion of a half-infinite chain of linear springs and masses under the assumption that the resolution of the identity corresponding to an appropriate self-adjoint operator is known. The solution agrees with the one found earlier by Law [7] by a very different technique. The final section presents a formulation in terms of the theory of linear operators of a solution to the equations of motion of a chain of linear springs and masses extending indefinitely in both directions--again under the assumption that the resolution of the identity corresponding to an appropriate self-adjoint operator is known. Physical symmetry about some point in the chain, which was assumed in the problem studied by Martens [8], is not required.

Summary of Some Pertinent Results from the Theory of Linear Operators

The results to be summarized concern linear operators whose domains are subsets of a real, separable Hilbert space H with inner product (.,.). The operators are not assumed to be bounded, and in referring to them the word *linear* is usually omitted. Some of the statements also apply to nonlinear operators, but no use is made of this fact.

Definition 1. An operator A whose domain D(A) is dense in H is called symmetric if and only if $(Ax,y) = (x,Ay)$ for all x and y in D(A).

Definition 2. An operator A is called closed if and only if, whenever a sequence $\{x_n\}$ in D(A) has the properties (i) $x_n \to x$ and (ii) $Ax_n \to y$ (convergence here means $\|x-x_n\| \to 0$), then x is in D(A) and $Ax = y$.

Definition 3. Let A be an operator which has a closed extension. Then the minimal closed extension of A is called the closure of A and is denoted $\bar{A}$ ($\bar{A}$ is minimal in the sense that if B is any closed extension of A, then B is an extension of $\bar{A}$). It can be shown that A may be defined alternatively as follows. The domain of $\bar{A}$ is the set of all f in H such that there exists a sequence $\{f_n\}$ in D(A) for which $f_n \to f$ and $\{Af_n\}$ converges. For such an f, $\bar{A}f = \lim Af_n$ (see Akhiezer and Glazman [2], Section 38).

Definition 4. Let A be an operator whose domain is dense in H. A new operator A^*, the adjoint of A, is defined as follows. The domain of A^* is the set of all y in H such that there exists some g in H such that $(Ax,y) = (x,g)$ for all x in D(A). The adjoint is defined by $A^*y = g$ (the fact that A^* is well-defined follows from the fact that D(A) is dense in H).

Note that $D(A^*)$ certainly contains the zero vector and that if A is symmetric, $D(A^*)$ contains D(A) and A^* extends A. Furthermore A^* is always a closed operator (this fact is an immediate consequence of the above definitions). Hence a symmetric operator always has closed extensions and thus has a closure.

Theorem 1. If an operator A has a closure $\bar{A}$, then $(\bar{A})^* = A^*$.

Proof. Since $\bar{A}$ extends A, it follows immediately that A^* extends $(\bar{A})^*$. It remains to show that $(\bar{A})^*$ extends A^*. Let x be in $D(A^*)$; then it must be shown that x is in $D((\bar{A})^*)$ and $(\bar{A})^*x = A^*x$. From the definition of the adjoint it follows that it is sufficient to show that $(\bar{A}y,x) = (y,A^*x)$ for each y in $D(\bar{A})$. Let y be in $D(\bar{A})$, and let $\{y_n\}$ be a sequence in D(A) such that $y_n \to y$ and $Ay_n \to \bar{A}y$. Then $(\bar{A}y,x) = \lim(Ay_n,x) = \lim(y_n,A^*x) = (y,A^*x)$.

Definition 4. An operator A is said to be self-adjoint if $A^* = A$.

Definition 5. A symmetric operator A is said to be positive if $(Af,f) \geq 0$ for all f in D(A).

Theorem 2. Let A be a positive, self-adjoint operator. Then there exists a unique family E_t of positive, self-adjoint operators with the following properties:

(i) $E_tE_s = E_u$ where $u = \min(s,t)$;

(ii) $\lim_{\varepsilon\to 0^+} E_{t-\varepsilon} = E_t$;

(iii) $\lim_{t\to 0} E_t = 0$ and $\lim_{t\to\infty} E_t = I$ where 0 and I are the zero and identity operators on H;

(iv) if f is in D(A) and g is in H, then $(Af,g) = \int_0^\infty t\,d(E_t f,g)$;

(v) if f is in D(A), then $\|Af\|^2 = \int_0^\infty t^2 d(E_t f,f)$;

(vi) for each f and g in H, $E_t f$ is in D(A) and

$$(AE_t f,g) = \int_0^t s\,d(E_s f,g).$$

Proof. For a proof of (i) through (v) and the first part of (vi) see reference [2], volume 2, pages 36-41. The proof of the second part of (vi) follows from (i) and (iv) since

$$(AE_t f,g) = \int_0^t s\,d(E_s E_t f,g)$$

$$= \int_0^t s\,d(E_s f,g).$$

Corollary. Let A be a positive, symmetric operator whose closure is self-adjoint. Then the conclusion of the above theorem holds for A, except that (vi) must be replaced by

(vi)' for each f and g in H, $E_t f$ is in $D(A^*)$ and

$$(A^* E_t f,g) = \int_0^\infty s\,d(E_s f,g).$$

Proof. Since $\bar{A}$ extends A, (i) through (v) clearly hold for A, and the remainder of the conclusion follows from the fact that $\bar{A} = (\bar{A})^* = A^*$.

Application of Operator Theory to the Solution of the Half-Infinite Linear Chain

Consider the chain of springs and masses shown in Figure 1a. The equations of motion for this chain are

$$m_0\ddot{x}_0 = -k_0x_0 + k_1(x_1-x_0),$$

$$m_n\ddot{x}_n = -k_n(x_n-x_{n-1}) + k_{n+1}(x_{n+1}-x_n), \qquad n\geq 1.$$

These equations may be written in the form

$$\ddot{x}_0 = -\frac{(k_0+k_1)x_0}{m_0} + \frac{k_1x_1}{m_0}.$$

$$\ddot{x}_n = \frac{k_nx_{n-1}}{m_n} - \frac{(k_n+k_{n+1})x_n}{m_n} + \frac{k_{n+1}x_{n+1}}{m_n}, \qquad n\geq 1$$

For convenience define $a_n = -k_{n+1}/\sqrt{m_nm_{n+1}}$, $n\geq 0$; $b_n = (k_n+k_{n+1})/m_n$, $n\geq 0$ (note that this notation is different from the one used by Law [7]). By putting $y_n = \sqrt{m_n/m_0}\ x_n$, $n\geq 0$, the system of equations may be written

$$\ddot{y}_0 = -(b_0y_0+a_0y_1), \tag{1}$$

$$\ddot{y}_n = -(a_{n-1}y_{n-1}+b_ny_n+a_ny_{n+1}), \qquad n\geq 1.$$

In order to formulate a solution to the system (1) by use of linear operator theory, a Hilbert space and an operator defined on it are introduced.

An appropriate Hilbert space is the set of all real-valued sequences $\{u_n\}_0^\infty$ such that $\sum_{n=0}^{\infty} u_n^2 < \infty$. If $u = \{u_n\}_0^\infty$ and $v = \{v_n\}_0^\infty$ are two such sequences, then $(u,v) = \sum_{n=0}^{\infty} u_n v_n$. This Hilbert space, which will be denoted $\ell_2(0,\infty)$, has a dense subspace $\ell_{2,0}(0,\infty)$ consisting of those sequences which have only finitely many non-zero terms.

An operator A is defined on $\ell_{2,0}(0,\infty)$ as follows:

$$(Au)_0 = b_0 u_0 + a_0 u_1,$$

$$(Au)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}, \qquad n \geq 1.$$

Since a_n $(n\geq 0)$ and b_n $(n\geq 0)$ are real, A maps $\ell_{2,0}(0,\infty)$ into itself. A is symmetric, since if u and v are in $\ell_{2,0}(0,\infty)$,

$$\begin{aligned}
(Au,v) &= \sum_{n=0}^{\infty} (Au)_n v_n \\
&= \sum_{n=1}^{\infty} a_{n-1}u_{n-1}v_n + \sum_{n=0}^{\infty} b_n u_n v_n + \sum_{n=0}^{\infty} a_n u_{n+1} v_n \\
&= \sum_{n=0}^{\infty} a_n u_n v_{n+1} + \sum_{n=0}^{\infty} b_n u_n v_n + \sum_{n=1}^{\infty} a_{n-1} u_n v_{n-1} \\
&= \sum_{n=0}^{\infty} (Av)_n u_n \\
&= (u,Av)
\end{aligned}$$

(since all the series are actually finite sums, no convergence questions are involved). A is also positive; for if f is in $\ell_{2,0}(0,\infty)$,

$$
\begin{aligned}
(Af,f) &= \sum_{n=0}^{\infty} (Af)_n f_n \\
&= \sum_{n=1}^{\infty} a_{n-1} f_{n-1} f_n + \sum_{n=0}^{\infty} b_n f_n^2 + \sum_{n=0}^{\infty} a_n f_{n+1} f_n \\
&= -\sum_{n=1}^{\infty} \frac{k_n f_{n-1} f_n}{\sqrt{m_{n-1} m_n}} + \sum_{n=0}^{\infty} \frac{(k_n + k_{n+1}) f_n^2}{m_n} - \sum_{n=0}^{\infty} \frac{k_{n+1} f_{n+1} f_n}{\sqrt{m_n m_{n+1}}} \\
&= \frac{k_0 f_0^2}{m_0} + \sum_{n=1}^{\infty} k_n \left(\frac{f_{n-1}}{\sqrt{m_{n-1}}} - \frac{f_n}{\sqrt{m_n}} \right)^2 \geq 0 .
\end{aligned}
$$

It will be assumed that the sequences $\{m_n\}$ and $\{k_n\}$ satisfy sufficient conditions to assure that the operator $\bar{A}$ is self-adjoint. One such sufficient condition is that the total mass $\sum_{n=0}^{\infty} m_n$ is infinite (see Appendix A). For other sufficient conditions see Berezanskii [4], Chapter VII, Theorems 1.2 and 1.3. This assumption having been made, it follows from the corollary to Theorem 2 (above) that there exists a unique resolution of the identity E_x with the following properties:

(i) $(Af,g) = \int_0^{\infty} x\,d(E_x f,g)$ for all f in $\ell_{2,0}(0,\infty)$ and g in $\ell_2(0,\infty)$;

(ii) $\|Af\|^2 = \int_0^{\infty} x^2 d(E_x f,f)$ for all f in $\ell_{2,0}(0,\infty)$;

(iii) for all f and g in $\ell_2(0,\infty)$, $E_x f$ is in $D(A^*)$ and

$$(A^* E_x f,g) = \int_0^x u\,d(E_u f,g).$$

To facilitate further study of A it is helpful to introduce the standard orthonormal basis of $\ell_2(0,\infty)$--that is, the set of vectors $\{e_0, e_1, e_2, \ldots\}$ where e_n is the vector whose nth coordinate is 1 and all of whose other coordinates are 0. Clearly e_n is in D(A) for each n; and

$$Ae_0 = a_0 e_1 + b_0 e_0,$$

$$Ae_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1}, \qquad n \geq 1.$$

These expressions for Ae_0 and Ae_n $(n \geq 1)$ make it possible to compute $A^k e_n$ for any $k \geq 0$ $(A^0 = I)$ and hence to compute $P(A)e_n$, where P is any polynomial with real coefficients. A sequence of theorems concerning polynomials in the operator A now leads to a formulation of a solution of the system of differential equations.

Theorem 3. Let $G_0(x) = 1$, $G_1(x) = \frac{1}{a_0}(x-b_0)$ and $a_n G_{n+1}(x) = (x-b_n)G_n(x) - a_{n-1}G_{n+1}(x)$, $n \geq 1$. Then for each $n \geq 0$, $G_n(A)e_0 = e_n$.

Proof. Since $G_0(A) = I$, $G_0(A)e_0 = e_0$; also $G_1(A)e_0 = \frac{1}{a_0}(A-b_0 I)e_0 = \frac{1}{a_0}(Ae_0 - b_0 e_0) = e_1$.

Thus the theorem is true for n = 0 and n = 1. Assume $e_{k-1} = G_{k-1}(A)e_0$ and $e_k = G_k(A)e_0$; then

$$G_{k+1}(A)e_0 = \frac{1}{a_k}[(A-b_kI)G_k(A)e_0-a_{k-1}G_{k-1}(A)e_0]$$

$$= \frac{1}{a_k}[a_ke_{k+1}+b_ke_k+a_{k-1}e_{k-1}-b_ke_k-a_{k-1}e_{k-1}]$$

$$= e_{k+1}.$$

Hence the theorem is true for every non-negative integer n.

Theorem 4. Suppose that P and Q are polynomials with real coefficients and that f and g are in $\ell_{2,0}(0,\infty)$. Then $(P(A)f,Q(A)g) = (P(A)Q(A)f,g)$.

Proof. Since A maps $\ell_{2,0}(0,\infty)$ into itself, P(A)f and Q(A)g are certainly defined and are in $\ell_{2,0}(0,\infty)$. Since the polynomials have real coefficients and the inner product is linear, it is sufficient to show that $(A^nf,A^mg) = (A^{n+m}f,g)$, $n,m\geq 0$. But $(A^nf,A^mg) = (A^nf,A(A^{m-1}g)) = (A^{n+1}f,A^{m-1}g) = (A^{n+2}f,A^{m-2}g) = \ldots = (A^{m+n}f,g)$, where the symmetry of A and the fact that A^kf and A^kg are in $\ell_{2,0}(0,\infty)$ have been used.

Theorem 5. If P is a polynomial with real coefficients and f and g are in $\ell_{2,0}(0,\infty)$, then $(P(A)f,g) = \int_0^\infty P(x)d(E_xf,g)$.

Proof. Again it is sufficient to establish that $(A^nf,g) = \int_0^\infty x^nd(E_xf,g)$ for $n\geq 0$.

This result is clearly true for n=0 and n=1. Assume that it is true for n=k, then

$$(A^{k+1}f,g) = (A^k f,Ag)$$

$$= \int_0^\infty x^k d(E_x f,Ag).$$

But

$$(E_x f,Ag) = (A^* E_x f,g)$$

$$= \int_0^x u d(E_u f,g).$$

Therefore $d(E_x f,Ag) = xd(E_x f,g)$, and hence

$$(A^{k+1}f,g) = \int_0^\infty x^{k+1} d(E_x f,g).$$

Theorem 6. Define $\alpha(x) = (E_x e_0,e_0)$. Then if $G_n(x)$ and $G_m(x)$ are defined as in Theorem 3,

$$\int_0^\infty G_n(x)G_m(x)d\alpha(x) = \delta_{m,n}, \qquad n,m\geq 0.$$

Proof.

$$\delta_{m,n} = (e_n,e_m)$$

$$= \left(G_n(A)e_0,G_m(A)e_0\right)$$

$$= \left(G_n(A)G_m(A)e_0,e_0\right)$$

$$= \int_0^\infty G_n(x)G_m(x)d\alpha(x)$$

A solution of the original system of differential equations subject to the initial conditions $x_k(0) = \alpha_k$, $\dot{x}_k(0) = \beta_k$, $x_n(0) = \dot{x}_n(0) = 0$, $n \neq k$, is now readily formulated. Since $x_n = \sqrt{m_0/m_n}\, y_n$, it is sufficient to find a solution of the system (1) subject to $y_k(0) = \sqrt{m_k/m_0}\, \alpha_k$, $\dot{y}_k(0) = \sqrt{m_k/m_0}\, \beta_k$, $y_n(0) = \dot{y}_n(0) = 0$, $n \neq k$.

Theorem 7. Suppose that for fixed real x, $F(x,t)$ is the unique solution to the differential equation $\ddot{F} = -xF$ which satisfies $F(x,0) = \sqrt{m_k/m_0}\, \alpha_k$ and $F_2(x,0) = \sqrt{m_k/m_0}\, \beta_k$. Then if

$$y_n(t) = \int_0^\infty G_n(x)G_k(x)F(x,t)d\alpha(x),$$

$y_n(t)$ satisfies the differential equations (1) and the initial conditions $y_k(0) = \sqrt{m_k/m_0}\, \alpha_k$, $\dot{y}_k(0) = \sqrt{m_k/m_0}\, \beta_k$, $y_n(0) = \dot{y}_n(0) = 0$, $n \neq k$.

Proof. In Appendix B it is shown that

$$\dot{y}_n(t) = \int_0^\infty G_n(x)G_k(x)F_2(x,t)d\alpha(x),$$

and

$$\ddot{y}_n(t) = \int_0^\infty G_n(x)G_k(x)F_{22}(x,t)d\alpha(x).$$

Hence Theorem 6 shows that the initial conditions are satisfied. The system of differential equations is satisfied because of the recurrence relation satisfied by the sequence of polynomials $\{G_n(x)\}_0^\infty$.

Applications of Operator Theory to the Fully Infinite Linear Chain

In this section techniques similar to those in the last section are used to formulate a solution of the equations of motion of a fully infinite chain of springs and masses, again under the assumption that the resolution of the identity of a certain self-adjoint operator is known. The application of techniques from operator theory produces an extension of previously known results--namely, a solution for a system that does not have the property of physical symmetry assumed by Martens [8].

Consider the chain of linear springs and masses extending indefinitely in either direction as shown in Figure 1b. The equations of motion of this chain are

$$m_n \ddot{x}_n = k_n(x_{n-1}-x_n) + k_{n+1}(x_{n+1}-x_n), \qquad n = \ldots,-1,0,1,\ldots,$$

or, upon division by m_n and rearrangement,

$$\ddot{x}_n = \frac{k_n x_{n-1}}{m_n} - \frac{(k_n+k_{n+1})x_n}{m_n} + \frac{k_{n+1}x_{n+1}}{m_n}, \qquad n = \ldots,-1,0,1,\ldots.$$

As in the previous section, it is convenient to make the preliminary transformation $x_n = \sqrt{m_0/m_n}\, y_n$. The system of equations may then be written

$$\ddot{y}_n = \frac{k_n y_{n-1}}{\sqrt{m_{n-1}m_n}} - \frac{(k_n+k_{n+1})y_n}{m_n} + \frac{k_{n+1}y_{n+1}}{\sqrt{m_n m_{n+1}}}, \qquad n = \ldots,-1,0,1,\ldots.$$

Let $a_n = -k_{n+1}/\sqrt{m_n m_{n+1}}$, $b_n = \dfrac{(k_{n+1}+k_n)}{m_n}$, $n = \ldots,-1,0,1,\ldots,$ so that

$$\ddot{y}_n = -(a_{n-1}y_{n-1}+b_n y_n+a_n y_{n+1}), \qquad n = \ldots,-1,0,1,\ldots. \tag{2}$$

An appropriate Hilbert space for this problem is the set of all real-valued sequences $\{u_n\}_{-\infty}^{\infty}$ such that $\sum_{-\infty}^{\infty} u_n^2 < \infty$. If $u = \{u_n\}_{-\infty}^{\infty}$ and $v = \{v_n\}_{-\infty}^{\infty}$ are two such sequences, then $(u,v) = \sum_{-\infty}^{\infty} u_n v_n$. This Hilbert space, which will be denoted $\ell_2(-\infty,\infty)$, has a dense subspace $\ell_{2,0}(-\infty,\infty)$ consisting of those sequences which have only finitely many non-zero terms.

An operator B is defined on $\ell_{2,0}(-\infty,\infty)$ as follows:

$$(Bu)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}, \qquad n = \ldots,-1,0,1,\ldots.$$

Clearly B maps $\ell_{2,0}(-\infty,\infty)$ into itself; and by arguments similar to the ones in the last section, B is symmetric and positive. It will again be assumed that $\{m_n\}_{-\infty}^{\infty}$ and $\{k_n\}_{-\infty}^{\infty}$ satisfy sufficient conditions to guarantee that $\bar{B}$ is self-adjoint. One such condition is that $\sum_{n=1}^{\infty} m_{-n}$ and $\sum_{n=0}^{\infty} m_n$ both diverge. Another sufficient condition will be found on page 584 of Berezanskii [4].

By the corollary to Theorem 2, there exists a unique resolution of the identity E_x with the following properties:

(i) $(Bf,g) = \int_0^{\infty} x\,d(E_x f,g)$ for all f in $\ell_{2,0}(-\infty,\infty)$ and g in $\ell_2(-\infty,\infty)$;

(ii) $\|Bf\|^2 = \int_0^\infty x^2 d(E_x f,f)$ for all f in $\ell_{2,0}(-\infty,\infty)$;

(iii) for all f and g in $\ell_2(-\infty,\infty)$, $E_x f$ is in $D(B^*)$ and

$$(B^* E_x f,g) = \int_0^x u d(E_u f,g).$$

The procedure used for formulating a solution of the system (2) is similar to that used in the previous section for formulating a solution to the system (1). Again the standard orthonormal basis $\{\ldots,e_{-1},e_0,e_1,\ldots\}$ is introduced and used to define polynomials in the operator B. A sequence of theorems concerning these operators is proved. Many of the proofs are omitted here since they are almost identical to the proofs of the corresponding theorems in the previous section.

Theorem 8. Let two sequences of polynomials $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ be defined by the recurrence relation

$$xT_n(x) = a_n T_{n+1}(x) + b_n T_n(x) + a_{n-1} T_{n-1}(x), \qquad n = \ldots,-1,0,1,\ldots,$$

and by the initial conditions $P_{-1}(x) = 0$, $P_0(x) = 1$, $Q_{-1}(x) = 1$, and $Q_0(x) = 0$. Then for each integer n

$$e_n = P_n(B)e_0 + Q_n(B)e_{-1}.$$

Proof. As in Theorem 3 the proof follows from the initial conditions and the recurrence relation by the use of mathematical induction.

Theorem 9. Suppose that P and Q are polynomials with real coefficients and that f and g are in $\ell_{2,0}(-\infty,\infty)$. Then

$$\bigl(P(B)f,Q(B)g\bigr) = \bigl(P(B)Q(B)f,g\bigr).$$

Proof. Similar to Theorem 4.

Theorem 10. If P is a polynomial with real coefficients and f and g are in $\ell_{2,0}(-\infty,\infty)$, then

$$\bigl(P(B)f,g\bigr) = \int_0^\infty P(x)d(E_x f,g).$$

Proof. Similar to Theorem 5.

Theorem 11. Define $\alpha_1(x) = (E_x e_0,e_0)$, $\alpha_2(x) = (E_x e_0,e_{-1})$, $\alpha_3(x) = (E_x e_{-1},e_0)$, and $\alpha_4(x) = (E_x e_{-1},e_{-1})$--note that since E_x is self-adjoint, $\alpha_2(x) = \alpha_3(x)$. Then

$$\int_0^\infty P_n(x)P_m(x)d\alpha_1(x) + \int_0^\infty P_n(x)Q_m(x)d\alpha_2(x)$$
$$+ \int_0^\infty P_m(x)Q_n(x)d\alpha_3(x)$$
$$+ \int_0^\infty Q_n(x)Q_m(x)d\alpha_4(x) = \delta_{m,n}.$$

Proof.

$$\begin{aligned}
\delta_{m,n} &= (e_m, e_n) \\
&= \left(P_m(B)e_0 + Q_m(B)e_{-1},\ P_n(B)e_0 + Q_n(B)e_{-1}\right) \\
&= \left(P_m(B)e_0, P_n(B)e_0\right) + \left(Q_m(B)e_{-1},\ P_n(B)e_0\right) \\
&\quad + \left(P_m(B)e_0,\ Q_n(B)e_{-1}\right) + \left(Q_n(B)e_{-1},\ Q_m(B)e_{-1}\right) \\
&= \left(P_m(B)P_n(B)e_0, e_0\right) + \left(Q_m(B)P_n(B)e_{-1}, e_0\right) \\
&\quad + \left(P_m(B)Q_n(B)e_0, e_{-1}\right) + \left(Q_m(B)Q_n(B)e_{-1}, e_{-1}\right) \\
&= \int_0^\infty P_n(x)P_m(x)\,d\alpha_1(x) \\
&\quad + \int_0^\infty P_n(x)Q_m(x)\,d\alpha_2(x) \\
&\quad + \int_0^\infty P_m(x)Q_n(x)\,d\alpha_3(x) \\
&\quad + \int_0^\infty Q_n(x)Q_m(x)\,d\alpha_4(x).
\end{aligned}$$

A solution to the equations of motion of the fully infinite linear chain, subject to the initial conditions $x_k(0) = \alpha_k$, $\dot{x}_k(0) = \beta_k$, $x_n(0) = \dot{x}_n(0) = 0$, $n \neq k$, may now be formulated. Since $y_n = \sqrt{m_o/m_n}\, x_n$, it is sufficient to solve the system (2) subject to the conditions $y_k(0) = \sqrt{m_k/m_0}\, \alpha_k$, $\dot{y}_k(0) = \sqrt{m_k/m_0}\, \beta_k$, $y_n(0) = \dot{y}_n(0) = 0$, $n \neq k$.

Theorem 12. Suppose that $F(x,t)$ is defined as in Theorem 7. Then

$$y_n(t) = \int_0^\infty P_n(x)P_k(x)F(x,t)d\alpha_1(x)$$

$$+ \int_0^\infty P_n(x)Q_k(x)F(x,t)d\alpha_2(x)$$

$$+ \int_0^\infty P_k(x)Q_n(x)F(x,t)d\alpha_3(x)$$

$$+ \int_0^\infty Q_n(x)Q_k(x)F(x,t)d\alpha_4(x)$$

is a solution of the system of equations (2) satisfying $y_k(0) = \sqrt{m_k/m_0}\ \alpha_k$, $\dot{y}_k(0) = \sqrt{m_k/m_0}\ \beta_k$, $y_n(0) = \dot{y}_n(0) = 0$, $n \neq k$.

Proof. Similar to Theorem 7.

CHAPTER III

SOME COMMENTS ON FINDING THE RESOLUTION OF THE IDENTITY FOR A SELF-ADJOINT OPERATOR

Solutions for the equations of motion of the half-infinite and fully infinite chains have been formulated in Chapter II under the assumption that the resolution of the identity for a given self-adjoint operator is known. The question of how to produce the resolution of the identity for such an operator arises naturally.

In its complete generality, this question is beyond the scope of the present study and perhaps beyond the limits of present mathematical knowledge. However, for the type of operator which is involved in the study of the half-infinite chain, finding the resolution of the identity is equivalent to finding an integrator for a sequence of orthogonal polynomials whose recurrence relation is known. Some progress has been made in certain special cases of the latter question (see Jayne [6], Law [7], and Martens [8]).

The goal of this chapter is to describe a method of finding the resoltuion of the identity for the type of operator involved in the study of the fully infinite chain. The method depends on being able to find integrators for two sequences of orthogonal polynomials whose recurrence relations are known.

In order to achieve this goal, it will be necessary to use some results from the theory of orthogonal polynomials which may be found in

Berezanskii [4], Chapter VII. Consider the recurrence relation

$$\alpha_n T_{n+1}(x) = (x-\beta_n)T_n(x) - \alpha_{n-1}T_{n-1}(x), \qquad n = 0,1,\ldots,$$

where $\alpha_n < 0$ and β_n is real for each n. Let $\{R_n(x)\}_{-1}^{\infty}$ be a sequence of polynomials satisfying this recurrence relation and the initial conditions $R_{-1}(x) = 0$, $R_0(x) = 1$. The R_n's are called polynomials of the first kind corresponding to the given recurrence relation. Let $\{S_n(x)\}_0^{\infty}$ be a sequence of polynomials satisfying the recurrence relation for $n \geq 1$ and the initial conditions $S_0(x) = 0$, $S_1(x) = \frac{1}{\alpha_0}$. The S_n's are called polynomials of the second kind. Suppose that there is exactly one real-valued, non-decreasing function $\sigma(x)$ having infinitely many points of increase, satisfying $\lim_{x\to-\infty} \sigma(x) = 0$, and such that

$$\int_{-\infty}^{\infty} R_n(x)R_m(x)d\sigma(x) = \delta_{m,n}.$$

Let $m(z) = \int_{-\infty}^{\infty} \frac{d\sigma(x)}{x - z}$, where $\mathrm{Im}(z) \neq 0$ $\left(\text{this integral exists because } \int_{-\infty}^{\infty} d\sigma(x) = 1\right)$. Then by the argument given in Section 1.6, Chapter VII, of Berezanskii, $\sum_{n=0}^{\infty} |S_n(z)+m(z)R_n(z)|^2 < \infty$ for each nonreal z; and if f(z) satisfies $\sum_{n=0}^{\infty} |S_n(z)+f(z)R_n(z)|^2 < \infty$ for each nonreal z, then $f(z) = m(z)$. The function m(z) is called the Stieltjes transform of $\sigma(x)$.

A similar result holds for matrix polynomials generated by recurrence relations whose coefficients are two-by-two matrices rather than real numbers--that is, for sequences $\{\hat{R}_n(x)\}_{-1}^{\infty}$ and $\{\hat{S}_n(x)\}_{-1}^{\infty}$ which satisfy

$$\hat{R}_{-1}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{R}_0(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\hat{\alpha}_n \hat{R}_{n+1}(x) = (xI-\hat{\beta}_n)\hat{R}_n(x) - \hat{\alpha}_{n-1}\hat{R}_{n-1}(x), \qquad n = 0,1,\ldots,$$

and

$$\hat{S}_0(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \hat{S}_1(x) = (\hat{\alpha}_0)^{-1},$$

$$\hat{\alpha}_n \hat{S}_{n+1}(x) = (xI-\hat{\beta}_n)\hat{S}_n(x) - \hat{\alpha}_{n-1}\hat{S}_{n-1}(x), \qquad n = 1,2,\ldots.$$

Here $\hat{\alpha}_n$ and $\hat{\beta}_n$ are self-adjoint two-by-two matrices and $\hat{\alpha}_n$ is invertible. The $\hat{R}_n$'s and $\hat{S}_n$'s are called matrix polynomials of the first and second kind, respectively. Suppose that there exists exactly one two-by-two matrix-valued function $\hat{\sigma}(x)$ such that

$$\int_{-\infty}^{\infty} [\hat{R}_m(x)d\hat{\sigma}(x)\hat{R}_n^*(x)] = \delta_{m,n}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(where $\hat{R}_n^*(x)$ is the conjugate transpose of $\hat{R}_n(x)$ and the integral of the matrix on the left side of this equation is interpreted as the matrix whose entries are the integrals of the entries of the integrand). Let $\hat{m}(z) = \int_{-\infty}^{\infty} \frac{d\hat{\sigma}(x)}{x - z}$ for each nonreal z. By the argument in Section 2.10, Chapter VII, of Berezanskii [4], m(z) is the unique matrix-valued function such that $\sum_{n=0}^{\infty} \left(\hat{S}_n(z)+\hat{m}(z)\hat{R}_n(z)\right)^*\left(\hat{S}_n(z)+\hat{m}(z)\hat{R}_n(z)\right) < \infty$ (the series convergence is interpreted as convergence of each of the series of entries of the matrix).

The above results may be used to describe a method for finding the integrators α_1, α_2, and α_4 which appeared in the formulation of a solution to the equations of motion of the fully infinite chain. Recall that the polynomials $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ satisfy $P_{-1}(x) = 0$, $P_0(x) = 1$,

$$a_n P_{n+1}(x) = (x-b_n)P_n(x) - a_{n-1}P_{n-1}(x), \qquad n = \ldots,-1,0,1,\ldots,$$

$Q_{-1}(x) = 1, \quad Q_0(x) = 0,$

$$a_n Q_{n+1}(x) = (x-b_n)Q_n(x) - a_{n-1}Q_{n-1}(x), \qquad n = \ldots,-1,0,1,\ldots,$$

and

$$\int_0^\infty P_n(x)P_m(x)d\alpha_1(x) + \int_0^\infty [P_n(x)Q_m(x)+P_m(x)Q_n(x)]d\alpha_2(x) + \int_0^\infty Q_n(x)Q_m(x)d\alpha_4(x) = \delta_{m,n}.$$

Four sequences of polynomials are defined in terms of the sequences $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ as follows:

$$R_n^+(x) = P_n(x), \qquad n = -1,0,\ldots,$$

$$S_n^+(x) = \frac{-1}{a_{-1}} Q_n(x), \qquad n = 0,1,\ldots,$$

$$R_n^-(x) = Q_{-n-1}(x), \qquad n = -1,0,\ldots,$$

$$S_n^-(x) = -\frac{1}{a_{-1}} P_{-n-1}(x), \qquad n = 0,1,\ldots.$$

It is easily seen that the R_n^+'s and S_n^+'s are polynomials of the first and second kinds, respectively, for the recurrence coefficients $\alpha_n = a_n$, $\beta_n = b_n$. Similarly the R_n^-'s and S_n^-'s are polynomials of the first and second kinds for the recurrence coefficients $\alpha_n = a_{-n-2}$, $\beta_n = b_{-n-1}$. The R_n^+'s and R_n^-'s are orthogonal polynomials (Law [7], p. 18); and if sufficient conditions are assumed on the a_n and b_n to make the operator $\tilde{B}$ of Chapter II self-adjoint, the sequences $\{R_n^+(x)\}_{-1}^{\infty}$ and $\{R_n^-(x)\}_{-1}^{\infty}$ have unique integrators $\sigma^+(x)$ and $\sigma^-(x)$ (see Appendix A). Let $m^+(z)$ and $m^-(z)$ be the Stieltjes transforms of $\sigma^+(z)$ and $\sigma^-(z)$.

Now define

$$\hat{R}_{-1}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\hat{R}_n(x) = \begin{bmatrix} Q_{-n-1}(x) & P_{-n-1}(x) \\ Q_n(x) & P_n(x) \end{bmatrix}, \qquad n = 0,1,\ldots,$$

and

$$\hat{S}_n(x) = -\frac{1}{a_{-1}} \begin{bmatrix} P_{-n-1}(x) & 0 \\ 0 & Q_n(x) \end{bmatrix}, \quad n = 0,1,\ldots.$$

It is easily verified that these sequences of matrix polynomials are polynomials of the first and second kind for the recurrence coefficients

$$\hat{\alpha}_n = \begin{bmatrix} a_{-n-2} & 0 \\ 0 & a_n \end{bmatrix}, \qquad n = 0,1,\ldots,$$

$$\hat{\beta}_0 = \begin{bmatrix} b_{-1} & a_{-1} \\ a_{-1} & b_0 \end{bmatrix}, \quad \text{and} \quad \hat{\beta}_n = \begin{bmatrix} b_{-n-1} & 0 \\ 0 & b_n \end{bmatrix}, \quad n = 1,2,\ldots,$$

and that the $\hat{R}_n$'s satisfy

$$\int_{-\infty}^{\infty} [\hat{R}_n(x)d\hat{\sigma}(x)\hat{R}_m^*(x)] = \delta_{m,n}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$\hat{\sigma}(x) = \begin{bmatrix} \alpha_4(x) & \alpha_2(x) \\ \alpha_2(x) & \alpha_1(x) \end{bmatrix} \quad \text{for} \quad x \geq 0$$

and

$$\hat{\sigma}(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{for} \quad x < 0$$

(the uniqueness of $\hat{\sigma}(x)$ follows from the fact that $\bar{B}$ is self-adjoint). Hence there exists a unique two-by-two matrix $\hat{m}(z)$ such that

$$\sum_{n=0}^{\infty} \left(\hat{S}_n(z)+\hat{R}_n(z)\hat{m}(z)\right)^*\left(\hat{S}_n(z)+\hat{R}_n(z)\hat{m}(z)\right) < \infty$$

for each nonreal z. But by writing

$$\hat{m}(z) = \begin{bmatrix} M_4(z) & M_2(z) \\ M_2(z) & M_1(z) \end{bmatrix}$$

and by using the fact that the convergence of the series of matrices is interpreted as convergence of the series of entries, it is easily seen that for each nonreal z

$$\{M_4(z)Q_{-n-1}(z) + \left[M_2(z) - \frac{1}{a_{-1}}\right]P_{-n-1}(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

$$\{M_2(z)Q_{-n-1}(z) + M_1(z)P_{-n-1}(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

$$\{M_4(z)Q_n(z) + M_2(z)P_n(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

and

$$\{\left[M_2(z) - \frac{1}{a_{-1}}\right]Q_n(z) + M_1(z)P_n(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty).$$

By virtue of the definitions of R_n^+, R_n^-, S_n^+, and S_n^-, the above statements imply that for each nonreal z

$$\{M_4(z)R_n^-(z) + \left[M_2(z) - \frac{1}{a_{-1}}\right]\left(-a_{-1}S_n^-(z)\right)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

$$\{M_2(z)R_n^-(z) + M_1(z)\left(-a_{-1}S_n^-(z)\right)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

$$\{M_4(z)\left(-a_{-1}S_n^+(z)\right) + M_2(z)R_n^+(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty);$$

and

$$\{\left[M_2(z) - \frac{1}{a_{-1}}\right]\left(-a_{-1}S_n^+(z)\right) + M_1(z)R_n^+(z)\}_0^\infty \qquad \text{is in } \ell_2(0,\infty).$$

Hence from the uniqueness of $m^+(z)$ and $m^-(z)$ it follows that

$$\frac{M_4(z)}{1 - a_{-1}M_2(z)} = m^-(z),$$

$$\frac{M_2(z)}{-a_{-1}M_1(z)} = m^-(z),$$

$$\frac{M_2(z)}{-a_{-1}M_4(z)} = m^+(z),$$

$$\frac{M_1(z)}{1 - a_{-1}M_2(z)} = m^+(z).$$

Solving these equations yields

$$M_1(z) = \frac{m^+(z)}{1 - a_{-1}^2 m^+(z)m^-(z)} ;$$

$$M_2(z) = \frac{-a_{-1}m^+(z)m^-(z)}{1 - a_{-1}^2 m^+(z)m^-(z)} ;$$

$$M_4(z) = \frac{m^-(z)}{1 - a_{-1}^2 m^+(z)m^-(z)} .$$

Hence if $\sigma^+(x)$ and $\sigma^-(x)$ are known, M_1, M_2, and M_4 may be computed from m^+ and m^-; and then α_1, α_2, and α_4 may be recovered from their Stieltjes transforms by the Stieltjes inversion formula (see Akhiezer [1], p. 124).

It should be emphasized that the method described for finding α_1, α_2, and α_4 depends on knowing σ^+ and σ^- and hence cannot be applied until the integrators of the sequences of orthogonal polynomials $\{R_n^+(x)\}_0^\infty$ and $\{R_n^-(x)\}_0^\infty$ have been found. Thus, a key step in the method described here for finding the resolution of the identity for the type of operator involved in the study of the fully infinite chain is that of finding integrators for the sequences $\{R_n^+(x)\}_0^\infty$ and $\{R_n^-(x)\}_0^\infty$.

CHAPTER IV

THE FULLY INFINITE UNIFORM LINEAR CHAIN

In this chapter the techniques of Chapters II and III are used to formulate a solution of the equations of motion of the fully infinite uniform linear chain--that is, the chain for which $m_n = m$ and $k_n = k$ for each n ($-\infty<n<\infty$). The representation of this solution is different from the one found by Martens [8] (see his Appendix B), but it is shown that the two solutions agree.

The recurrence relation satisfied by the sequences $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ is

$$T_{n+1}(x) = \left(-\frac{mx}{k} + 2\right)T_n(x) - T_{n-1}(x), \qquad n = \ldots,-1,0,1,\ldots.$$

By using this recurrence relation, the initial conditions, the substitution $-\frac{mx}{k} + 2 = -2\cos\theta$, and standard difference-equation techniques, it may be shown that $P_n[x(\theta)] = \frac{(-1)^n\sin(n+1)\theta}{\sin\theta}$ and $Q_n[x(\theta)] = \frac{(-1)^n\sin n\theta}{\sin\theta}$. Hence in this case $R_n^+[x(\theta)] = R_n^-[x(\theta)] = \frac{(-1)^n\sin(n+1)\theta}{\sin\theta}$. But it is known (see Szegö [9], page 60) that

$$\frac{\sin(n+1)\theta}{\sin\theta} = \frac{1}{2}\left[\frac{(2)(4)\ \ldots\ (2n+2)}{(1)(3)\ \ldots\ (2n+1)}\right]P_n^{(\frac{1}{2},\frac{1}{2})}(\cos\theta),$$

where $\{P_n^{(\frac{1}{2},\frac{1}{2})}(x)\}$ is the sequence of Jacobi polynomials orthogonal on

$[-1,1]$ with respect to the weight function $\sqrt{1-x^2}$. Since $\cos\theta = \frac{mx}{2k} - 1$, it follows that

$$d\sigma^+(x) = d\sigma^-(x) = \begin{cases} \dfrac{m^2}{2k^2\pi}\sqrt{x\left(\dfrac{4k}{m} - x\right)}\,dx, & 0\le x\le\dfrac{4k}{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$m^+(z) = m^-(z) = \frac{m^2}{2k^2\pi}\int_0^{4k/m} \frac{1}{x - z}\sqrt{x\left(\frac{4k}{m} - x\right)}\,dx.$$

This integral may be evaluated by making the substitution

$$u = \left[\frac{mx}{2k}\left(2 - \frac{mx}{2k}\right)^{-1}\right]^{1/2}$$

and using residue theory. The result is

$$m^+(z) = m^-(z) = \frac{m}{k}\left[-\left(\frac{m}{2k}z - 1\right) + i\left(2 - \frac{m}{2k}z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right],$$

where $\sqrt{}$ indicates that square root whose imaginary part is non-negative. Then a tedious computation shows that

$$M_1(z) = M_4(z) = \frac{-\left(\frac{m}{2k}z - 1\right) + i\left(2 - \frac{m}{2k}z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}}{\left(\frac{4k}{m} - z\right)\left[\frac{m}{2k}z + i\left(\frac{m}{2k}z - 1\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]},$$

$$M_2(z) = \frac{\frac{m^2}{2k^2} z^2 - \frac{2m}{k} z + 1 + i\, \frac{m}{k}\left(\frac{m}{2k} z - 1\right)\left(\frac{4k}{m} - z\right) \sqrt{\frac{z}{\frac{4k}{m} - z}}}{\left(\frac{4k}{m} - z\right)\left[\frac{m}{2k} z + i\left(\frac{m}{2k} z - 1\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]} .$$

By the Stieltjes inversion formula,

$$d\alpha_1(x) = d\alpha_4(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{x\left(\frac{4k}{m} - x\right)}}\, dx, & 0 \le x \le \frac{4k}{m} , \\ 0 & \text{otherwise;} \end{cases}$$

$$d\alpha_2(x) = \begin{cases} -\frac{1}{\pi} \frac{\left(\frac{m}{2k} x - 1\right)}{\sqrt{x\left(\frac{4k}{m} - x\right)}}\, dx, & 0 \le x \le \frac{4k}{m} , \\ 0 & \text{otherwise.} \end{cases}$$

Thus the solution of the equations of motion of the uniform linear chain, satisfying the initial conditions $x_k(0) = \alpha_k$, $\dot{x}_k(0) = \beta_k$, $x_n(0) = \dot{x}_n(0) = 0$, $n \ne k$, is

$$x_n(t) = \int_0^{4k/m} P_n(x) P_m(x) F(x,t)\, d\alpha_1(x)$$

$$+ \int_0^{4k/m} [P_n(x)Q_m(x) + P_m(x)Q_n(x)] F(x,t)\, d\alpha_2(x)$$

$$+ \int_0^{4k/m} Q_n(x)Q_m(x)F(x,t)d\alpha_4(x),$$

where $F(x,t)$ satisfies $F_{22}(x,t) = -xF(x,t)$, $F(x,0) = \alpha_k$, $F_2(x,0) = \beta_k$.

In order to compare this solution to the one obtained by Martens, it is convenient to make the change of variable $\cos\theta = \frac{mx}{2k} - 1$ in both solutions. The solution above may be written

$$x_n(t) = \frac{(-1)^{n+k}}{\pi} \int_0^{\pi} \frac{\sin(n+1)\theta\sin(k+1)\theta G(\theta,t)}{\sin^2\theta}\, d\theta$$

$$- \frac{(-1)^{n+k}}{\pi} \int_0^{\pi} \left[\frac{\sin(n+1)\theta\sin k\theta+\sin n\theta\sin(k+1)\theta}{\sin^2\theta}\right] G(\theta,t)\cos\theta d\theta$$

$$+ \frac{(-1)^{n+k}}{\pi} \int_0^{\pi} \frac{\sin n\theta\sin k\theta}{\sin^2\theta}\, G(\theta,t)d\theta,$$

where $G(\theta,t) = F\left(\frac{2k}{m}(1+\cos\theta),t\right)$. Martens' solution may be written in trigonometric form as

$$x_n(t) = \frac{(-1)^{n+k}}{\pi} \int_0^{\pi} \cos n\theta\cos k\theta G(\theta,t)d\theta$$

$$+ \frac{(-1)^{n+k}}{\pi} \int_0^{\pi} \sin n\theta\sin k\theta G(\theta,t)d\theta,$$

where $G(\theta,t)$ is the same as above. Combining the first two integrals and the last two integrals in the first solution yields

$$x_n(t) = \frac{(-1)^{n+k}}{\pi} \int_0^\pi \frac{\sin(n+1)\theta}{\sin^2\theta} [\sin(k+1)\theta - \sin k\theta \cos\theta] G(\theta,t)d\theta$$

$$+ \frac{(-1)^{n+k}}{\pi} \int_0^\pi \frac{\sin n\theta}{\sin^2\theta} [\sin(k+1-1)\theta - \sin(k+1)\theta\cos\theta] G(\theta,t)d\theta.$$

After simplification of the terms in square brackets,

$$x_n(t) = \frac{(-1)^{n+k}}{\pi} \int_0^\pi \left[\frac{\sin(n+1)\theta\cos k\theta - \sin n\theta\cos(k+1)\theta}{\sin\theta}\right] G(\theta,t)d\theta.$$

Expanding $\sin(n+1)\theta$ and $\cos(k+1)\theta$ then yields

$$x_n(t) = \frac{(-1)^{n+k}}{\pi} \int_0^\pi [\cos n\theta\cos k\theta + \sin n\theta \sin k\theta] G(\theta,t)d\theta,$$

and thus the solutions agree.

CHAPTER V

A FULLY INFINITE LINEAR CHAIN WITHOUT PHYSICAL SYMMETRY

In this chapter a solution is found for the equations of motion of a fully infinite chain not possessing physical symmetry about some middle mass. The lack of physical symmetry prevents the use of techniques employed by Martens [8]. However, a solution is found by the methods of Chapters II and III with no difficulty other than computational drudgery. Again it is assumed that appropriate integrators are known in advance for two sequences of orthogonal polynomials.

The chain to be studied has

$$m_n = \begin{cases} m \ , & n\le 0 \\ (n+1)^2 m, & n\ge 1 \end{cases} ; \qquad k_n = \begin{cases} k \ , & n\le 0, \\ n(n+1)k, & n\ge 1. \end{cases}$$

Hence $a_n = -\frac{k}{m}$, $n = \ldots,-1,0,1,\ldots$; $b_0 = \frac{3k}{m}$; and $b_n = \frac{2k}{m}$ for $n\neq 0$. Thus the recurrence relation satisfied by $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ is

$$T_{n+1}(x) = \left(-\frac{mx}{k} + 2\right)T_n(x) - T_{n-1}(x), \qquad n\neq 0,$$

$$T_1(x) = \left(-\frac{mx}{k} + 3\right)T_0(x) - T_{-1}(x).$$

By using this recurrence relation, the initial conditions, the substitution $-\frac{mx}{2k} + 2 = -2\cos\theta$, and standard difference-equation techniques, it can be shown that

$$P_n[x(\theta)] = \begin{cases} \dfrac{(-1)^n \cos(n+\frac{1}{2})\theta}{\cos\frac{\theta}{2}}, & n \geq 0 \\[2ex] \dfrac{(-1)^n \sin(n+1)\theta}{\sin\theta}, & n < 0 \end{cases};$$

$$Q_n[x(\theta)] = \frac{(-1)^n \sin n\theta}{\sin\theta}, \qquad n = \ldots,-1,0,1,\ldots.$$

The R_n^+'s and R_n^-'s are again seen to be Jacobi polynomials (Szegö [9], p. 60). Furthermore

$$d\sigma^+(x) = \begin{cases} \dfrac{m}{2k\pi}\sqrt{\dfrac{x}{\frac{4k}{m} - x}}\,dx, & 0 \leq x \leq \dfrac{4k}{m}, \\[2ex] 0 \quad \text{otherwise}; \end{cases}$$

and

$$d\sigma^-(x) = \begin{cases} \dfrac{m^2}{2k^2\pi}\sqrt{x\left(\dfrac{4k}{m} - x\right)}\,dx, & 0 \leq x \leq \dfrac{4k}{m}, \\[2ex] 0 \quad \text{otherwise}. \end{cases}$$

Hence

$$m^+(z) = \frac{m}{2k\pi}\int_0^{4k/m} \frac{1}{x - z}\sqrt{\frac{x}{\frac{4k}{m} - x}}\,dx, \qquad \mathrm{Im}(z) \neq 0,$$

and

$$m^-(z) = \frac{m^2}{2k^2\pi} \int_0^{4k/m} \frac{1}{x - z} \sqrt{x\left(\frac{4k}{m} - x\right)}\, dx, \qquad Im(z) \neq 0.$$

These integrals may be evaluated by the same method used in Chapter IV. The results are

$$m^+(z) = \frac{m}{2k}\left[1 + i\sqrt{\frac{z}{\frac{4k}{m} - z}}\right],$$

$$m^-(z) = \frac{m}{k}\left[-\left(\frac{m}{2k} z - 1\right) + i\left(2 - \frac{m}{2k} z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]$$

where the square root takes its values in the upper half-plane. Another tedious computation shows that

$$M_1(z) = \frac{\frac{m}{2k}\left[1 + i\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]}{\left(\frac{m}{2k} z + \frac{1}{2}\right) - i\left(\frac{3}{2} - \frac{m}{2k} z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}},$$

$$M_2(z) = \frac{-\frac{1}{2}\left[\left(-\frac{m}{k} z + 1\right) + i\left(3 - \frac{m}{k} z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]}{\left(\frac{m}{2k} z + \frac{1}{2}\right) - i\left(\frac{3}{2} - \frac{m}{2k} z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}},$$

$$M_4(z) = \frac{\frac{m}{k}\left[-\left(\frac{m}{2k}z - 1\right) + i\left(2 - \frac{m}{k}z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}\right]}{\left(\frac{m}{2k}z + \frac{1}{2}\right) - i\left(\frac{3}{2} - \frac{m}{2k}z\right)\sqrt{\frac{z}{\frac{4k}{m} - z}}}.$$

The application of the Stieltjes inversion formula is made more difficult in this case by the fact that each of the above functions has a pole at $z = \frac{k}{m}(2 + \sqrt{5})$. However, a technique due to Chihara [5] may be used to overcome the difficulty. A detailed account of the procedure is included as Appendix C. The results of the inversion are

$$d\alpha_1(x) = \left[\frac{1}{\pi} H(x)H\left(\frac{4k}{m} - x\right)\left(\frac{-\sqrt{x\left(\frac{4k}{m} - x\right)}}{x^2 - \frac{4k}{m}x - \frac{k^2}{m^2}}\right) + \frac{1}{\sqrt{5}}\,\delta\left(x - \frac{k}{m}(2 + \sqrt{5})\right)\right]dx,$$

$$d\alpha_2(x) = \left[\frac{1}{\pi} H(x)H\left(\frac{4k}{m} - x\right)\left(\frac{m}{2k}x - \frac{3}{2}\right)\left(\frac{\sqrt{x\left(\frac{4k}{m} - x\right)}}{x^2 - \frac{4k}{m}x - \frac{k^2}{m^2}}\right) + \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)\delta\left(x - \frac{k}{m}(2+\sqrt{5})\right)\right]dx,$$

$$d\alpha_3(x) = \left[\frac{1}{\pi} H(x)H\left(\frac{4k}{m} - x\right)\left(\frac{m}{2k} x - \frac{5}{2}\right)\left(\frac{\sqrt{x\left(\frac{4k}{m} - x\right)}}{x^2 - \frac{4k}{m} x - \frac{k^2}{m^2}}\right) + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 \delta\left(x - \frac{k}{m}(2+\sqrt{5})\right)\right] dx,$$

where H is the Heaviside unit step function and δ is the Dirac delta function. In Appendix D it is verified directly that these integrators provide the desired orthogonality relations.

Finally the solution of the equations of motion satisfying the initial conditions $x_k(0) = \alpha_k$, $\dot{x}_k(0) = \beta_k$, $x_n(0) = \dot{x}_n(0) = 0$, $n \neq k$, is

$$x_n(t) = \sqrt{m_k/m_n} \int_0^\infty P_n(x)P_k(x)F(x,t)d\alpha_1(x)$$

$$+ \sqrt{m_k/m_n} \int_0^\infty [P_n(x)Q_k(x)+Q_n(x)P_k(x)]F(x,t)d\alpha_2(x)$$

$$+ \sqrt{m_k/m_n} \int_0^\infty Q_n(x)Q_k(x)F(x,t)d\alpha_4(x),$$

where $F(x,t)$ satisfies $F_{22}(x,t) = -xF(x,t)$, $F(x,0) = \alpha_k$ and $F_2(x,0) = \beta_k$.

APPENDIX A

A SUFFICIENT CONDITION FOR THE SELF-ADJOINTNESS OF $\bar{A}$ AND $\bar{B}$

The following observation will be used in deriving a sufficient condition for the self-adjointness of $\bar{A}$ and $\bar{B}$: if T is a symmetric, positive operator, then $\bar{T}$ is self-adjoint if and only if T^* has no eigenvalue with nonzero imaginary part. The proof of this observation follows from propositions 1° and 2° on page 93 and Theorem 3 on page 97 of Akhiezer and Glazman [2], Volume II.

Consider the operator A. From the definition of adjoint it follows that if y is in $\ell_{2,0}(0,\infty)$, then y is in $D(A^*)$ if and only if $(Ax,y) = (x,A^*y)$ for each x in D(A). By letting $x = e_n$ (the vector whose nth coordinate is one and whose other coordinates are zero) and by recalling the definition of A from Chapter II, it is easily seen that

$$(A^*y)_0 = b_0y_0 + a_0y_1,$$

$$(A^*y)_n = a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1}, \qquad n = 1,2,\ldots.$$

Hence if $A^*y = zy$ for some complex number z with $\mathrm{Im}(z) \neq 0$ and some y in $\ell_2(0,\infty)$, then

$$zy_0 = b_0y_0 + a_0y_1,$$

$$zy_n = a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1}, \qquad n = 1,2,\ldots.$$

The solution of this boundary-value problem takes the form $y_n = cG_n(z)$, where $\{G_n(z)\}_0^\infty$ is the sequence of polynomials defined in Theorem 3 in the second section of Chapter II. Hence A^* has a nonreal eigenvalue if and only if $\{G_n(z)\}_0^\infty$ is in $\ell_2(0,\infty)$ for some nonreal z. But it is well known that if there exists a single nonreal z for which $\{G_n(z)\}_0^\infty$ is in $\ell_2(0,\infty)$, then $\{G_n(z)\}_0^\infty$ is in $\ell_2(0,\infty)$ for every nonreal z (Akhiezer and Glazman [2], vol. 2, p. 92). Hence to show that $\bar{A}$ is self-adjoint it is sufficient to show that $\sum_{n=0}^\infty |P_n(z)|^2$ diverges for some nonreal z. But since for $y \neq 0$ $|G_n(x+iy)| \geq |G_n(x)|$ (see Berezanskii [4], p. 521), it suffices to prove that $\sum_{n=0}^\infty \left(G_n(x)\right)^2$ diverges for some real x. It will now be shown that if $\sum_{n=0}^\infty m_n$ diverges then so does $\sum_{n=0}^\infty \left(G_n(0)\right)^2$. Recall that

$$a_n = \frac{-k_{n+1}}{\sqrt{m_nm_{n+1}}}, \qquad b_n = \frac{k_n + k_{n+1}}{m_n}.$$

Hence with x = 0 the recurrence relation for the G_n's becomes

$$\frac{k_{n+1}}{\sqrt{m_nm_{n+1}}}G_{n+1}(0) = \left(\frac{k_n+k_{n+1}}{m_n}\right)G_n(0) - \frac{k_n}{\sqrt{m_nm_{n-1}}}G_{n-1}(0), \qquad n = 1,2,\ldots.$$

Multiplying by $\sqrt{m_n}$ and transposing terms yields

$$k_{n+1}\left(\frac{G_{n+1}(0)}{\sqrt{m_{n+1}}} - \frac{G_n(0)}{\sqrt{m_n}}\right) = k_n\left(\frac{G_n(0)}{\sqrt{m_n}} - \frac{G_{n-1}(0)}{\sqrt{m_{n-1}}}\right), \qquad n = 1,2,\ldots.$$

Hence it is apparent that

$$k_j\left(\frac{G_j(0)}{\sqrt{m_j}} - \frac{G_{j-1}(0)}{\sqrt{m_{j-1}}}\right) = \frac{k_0}{\sqrt{m_0}}, \qquad j = 1,2,\ldots,$$

or

$$\sum_{j=1}^{n}\left(\frac{G_j(0)}{\sqrt{m_j}} - \frac{G_{j-1}(0)}{\sqrt{m_{j-1}}}\right) = \sum_{j=1}^{n}\frac{1}{k_j}\left(\frac{k_0}{\sqrt{m_0}}\right),$$

or

$$\frac{G_n(0)}{\sqrt{m_n}} = \frac{1}{\sqrt{m_0}}\left(1 + \sum_{j=1}^{n}\frac{k_0}{k_j}\right), \qquad n = 1,2,\ldots.$$

Clearly then $\left(G_n(0)\right)^2 \geq \frac{m_n}{m_0}$; and so if $\sum_{n=0}^{\infty} m_n$ diverges, so does $\sum_{n=0}^{\infty}\left(G_n(0)\right)^2$.

In order to show that $\tilde{B}$ is self-adjoint a procedure similar to the above is followed. As before it may be shown that if u is in $D(B^*)$,

$$(B^*u)_n = a_n u_{n+1} + b_n u_n + a_{n-1}u_{n-1}, \qquad n = \ldots,-1,0,1,\ldots.$$

Hence if $B^*y = zy$ for some z with $\mathrm{Im}(z) \neq 0$ and some y in $\ell_2(-\infty,\infty)$, then

$$zy_n = a_{n-1}y_{n-1} + b_ny_n + a_ny_{n+1}, \qquad n = \ldots,-1,0,1,\ldots.$$

The general solution of this difference equation takes the form $y_n = c_1P_n(z) + c_2Q_n(z)$, where $\{P_n(z)\}_{-\infty}^{\infty}$ and $\{Q_n(z)\}_{-\infty}^{\infty}$ are the linearly independent sequences of polynomials defined in the last section of Chapter II. Hence z is an eigenvalue of B^* if and only if for some choice of c_1 and c_2, $\{c_1P_n(z)+c_2Q_n(z)\}_{-\infty}^{\infty}$ is in $\ell_2(-\infty,\infty)$. It will now be shown that if $\sum_{n=1}^{\infty} m_{-n}$ and $\sum_{n=0}^{\infty} m_n$ diverge, then $\{c_1P_n(z)+c_2Q_n(z)\}_{-\infty}^{\infty}$ is not in $\ell_2(-\infty,\infty)$ for any choice of c_1 and c_2. The proof is by contradiction. If $\sum_{-\infty}^{\infty} |c_1P_n(z)+c_2Q_n(z)|^2 < \infty$, then certainly $\sum_{n=0}^{\infty} |c_1P_n(z)+c_2Q_n(z)|^2 < \infty$ and $\sum_{n=1}^{\infty} |c_1P_{-n}(z)+c_2Q_{-n}(z)|^2 < \infty$. But from Chapter III $R_n^+(z) = P_n(z)$, $n = -1,0,\ldots$; $S_n^+(z) = \frac{-1}{a_{-1}} Q_n(z)$, $n = 0,1,\ldots$; $R_n^-(z) = Q_{-n-1}(z)$, $n = -1,0,\ldots$; and $S_n^-(z) = \frac{-1}{a_{-1}} P_{-n-1}(z)$, $n = 0,1,\ldots$; so $\sum_{n=0}^{\infty} |c_1R_n^+(z)-a_{-1}c_2S_n^+(z)|^2 < \infty$ and $\sum_{n=0}^{\infty} |-a_{-1}c_1S_n^-(z)+c_2R_n^-(z)|^2 < \infty$. However, by the same argument as before, the divergence of $\sum_{n-1}^{\infty} m_{-n}$ and $\sum_{n=0}^{\infty} m_n$ implies the divergence of $\sum_{n=0}^{\infty} [R_n^+(0)]^2$ and $\sum_{n=0}^{\infty} [R_n^-(0)]^2$ and hence the divergence of $\sum_{n=0}^{\infty} [R_n^+(z)]^2$ and $\sum_{n=0}^{\infty} [R_n^-(z)]^2$ for all nonreal z. Hence the R_n^+'s and R_n^-'s have unique Stieltjes transforms. Thus the convergence of $\sum_{-\infty}^{\infty} |c_1P_n(z)+c_2Q_n(z)|^2$ (note that this series cannot converge if $c_1 = 0$ or $c_2 = 0$) implies that $\frac{c_1}{-a_{-1}c_2} = m^+(z)$ and $\frac{c_2}{-a_{-1}c_1} = m^-(z)$ and hence $a_{-1}^2 m^+(z)m^-(z) = 1$. But if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}\big(m^+(z)\big) > 0$, $\mathrm{Im}\big(m^-(z)\big) > 0$, and the above equation cannot be satisfied. Thus B^* has no eigenvalue with nonzero imaginary part, and so $\bar{B}$ is self-adjoint.

APPENDIX B

VALIDITY OF DIFFERENTIATION UNDER THE INTEGRAL SIGN

The purpose of this appendix is to verify two assertions made in Chapter II:

(1) if $$x_n(t) = \int_0^\infty G_n(x)G_m(x)F(x,t)d\alpha(x),$$

then $$x_n'(t) = \int_0^\infty G_n(x)G_m(x)F_2(x,t)d\alpha(x)$$

and $$x_n''(t) = \int_0^\infty G_n(x)G_m(x)F_{22}(x,t)d\alpha(x);$$

(2) if $$x_n(t) = \int_0^\infty P_n(x)P_m(x)F(x,t)d\alpha_1(x)$$

$$+ \int_0^\infty [P_n(x)Q_m(x)+P_m(x)Q_n(x)]F(x,t)d\alpha_2(x)$$

$$+ \int_0^\infty Q_n(x)Q_m(x)F(x,t)d\alpha_4(x),$$

then again $x_n'(t)$ and $x_n''(t)$ can be computed by differentiation through the integral sign. In order to simplify the verification of these assertions it is convenient to note some similarities of the integrators $\alpha(x)$, $\alpha_1(x)$, $\alpha_2(x)$ and $\alpha_4(x)$. Recall that $\alpha(x)$ is defined by the expression $\alpha(x) = (E_x e_0, e_0)$, where E_x is the resolution of the identity corresponding to $\bar{A}$. Recall also that if s and t are nonnegative real numbers, then $E_sE_t = E_u$ where $u = \min\{s,t\}$. In particular $E_s^2 = E_sE_s = E_s$. Thus if $s \geq t, (E_s-E_t)^2 = E_s^2 - E_sE_t - E_tE_s + E_t^2 = E_s - 2E_t + E_t = E_s - E_t$. Hence $\left((E_s-E_t)e_0,e_0\right) = \left((E_s-E_t)^2e_0,e_0\right) = \left((E_s-E_t)e_0,(E_s-E_t)e_0\right) \geq 0$ (the fact that E_x is self-adjoint for each x is used to justify the next-to-last step). Therefore $\alpha(s) \geq \alpha(t)$; that is, α is nondecreasing. It is also easily verified that $\alpha_1(x)$ and $\alpha_4(x)$ are nondecreasing by recalling that $\alpha_1(x) = (E_xe_0,e_0)$ and $\alpha_4(x) = (E_xe_{-1},e_{-1})$ where E_x is the resolution of the identity corresponding to $\bar{B}$. Since $\alpha_2(x) = (E_xe_0,e_{-1})$, it is not nondecreasing; but it may be written as the difference of nondecreasing functions by noting that

$$2(E_xe_0,e_{-1}) = \left(E_x(e_0+e_{-1}),e_0+e_{-1}\right) - (E_xe_0,e_0) - (E_xe_{-1},e_{-1}).$$

For convenience let $\psi(x) = \left(E_x(e_0+e_{-1}),e_0+e_{-1}\right)$. Then $\alpha_2(x) = \frac{1}{2}[\psi(x)-\alpha_1(x)-\alpha_4(x)]$.

The other important property possessed by $\alpha(x)$, $\alpha_1(x)$, $\alpha_2(x)$, and $\alpha_4(x)$ (and hence by $\psi(x)$) is that any polynomial is absolutely integrable on $[0,\infty)$ with respect to any of these functions. To verify this statement note that G_n is integrable with respect to α for each n, because $\int_0^\infty G_n(x)d\alpha(x) = \delta_{0,n}$. Since any polynomial may be written as a finite linear combination of the G_n's, any polynomial is integrable with respect to α on $[0,\infty)$. But if c is the largest positive zero of a polynomial P, then for $x > c$, $|P(x)| = P(x)$ or $|P(x)| = -P(x)$; so $|P|$

is integrable with respect to α. A similar proof shows that polynomials are absolutely integrable with respect to $\alpha_1(x)$, $\alpha_2(x)$, and $\alpha_4(x)$ $\{$and hence $\psi(x)\}$.

To complete the verification of assertions (1) and (2), it is only necessary to note some properties of $F(x,t)$. From the definition of $F(x,t)$ in Theorem 7 of Chapter II, it is easily seen that

$$F(x,t) = a_k \cos(\sqrt{x}\, t) + \frac{b_k}{\sqrt{x}} \sin(\sqrt{x}\, t),$$

$$F_2(x,t) = -a_k \sqrt{x} \sin(\sqrt{x}\, t) + b_k \cos(\sqrt{x}\, t),$$

and

$$F_{22}(x,t) = -a_k x \cos(\sqrt{x}\, t) + b_k \sqrt{x} \sin(\sqrt{x}\, t).$$

Clearly F, F_2, and F_{22} are continuous in the second variable for all t. Now since $x_n(t) = \int_0^\infty G_n(x)G_k(x)F(x,t)d\alpha(x)$

$$\begin{aligned} x_n(t) &= \int_0^\infty G_n(x)G_k(x)F(x,t)d\alpha(x) \\ &= \int_0^1 G_n(x)G_k(x)F(x,t)d\alpha(x) \\ &\quad + \int_1^\infty G_n(x)G_k(x)F(x,t)d\alpha(x), \end{aligned}$$

and since it has been previously verified that differentiation through the integral sign is valid on a finite interval (see Law [7], p. 30), in order to get the desired result it is only necessary to observe that for $x \geq 1$, $|F(x,t)| \leq |a_k| + |b_k|$, $|F_2(x,t)| \leq |a_k|x + |b_k|$, and $|F_{22}(x,t)| \leq (|a_k| + |b_k|)x$ and then to use Theorem 14-24 of Apostol [3] (the Weierstrass M-test shows that the relevant integrals converge uniformly). The second assertion follows by a similar argument.

APPENDIX C

RECOVERY OF THE INTEGRATORS α_1, α_2, AND α_4 FROM THEIR STIELTJES TRANSFORMS

In Chapter III it is shown that the Stieltjes transforms M_1, M_2, and M_4 of α_1, α_2, and α_4 can be expressed in terms of the Stieltjes transforms m^+ and m^-. It is then stated that if m^+ and m^- are known, α_1, α_2, and α_4 can be recovered from their Stieltjes transforms by the Stieltjes inversion formula. The purpose of this appendix is to give a more detailed explanation of how this process is carried out. Only one integrator α_1 will be considered since the procedure for the others is similar.

The Stieltjes inversion formula states that

$$\alpha_1(x) - \alpha_1(x_0) = \lim_{v\to 0} \frac{1}{\pi} \int_{x_0}^{x} \mathrm{Im}[M_1(u+iv)]du,$$

assuming that the value of α_1 at each of its at most countably many jumps is redefined as $\alpha_1(x) = \dfrac{\alpha_1(x+0) + \alpha_1(x-0)}{2}$ (see Akhiezer [1], p. 125). Since $m^+(z)$ and $m^-(z)$ are real-valued for any real z for which they are defined, the formula

$$M_1(z) = \frac{m^+(z)}{1 - a_{-1}^2 m^+(z)m^-(z)}$$

shows that $M_1(z)$ is real-valued for any real z for which m^+ and m^- are defined and $1 - a_{-1}^2m^+(z)m^-(z) \neq 0$. Hence in any real interval in which m^+ and m^- are defined and $1 - a_{-1}^2m^+(z)m^-(z)$ is non-zero, α_1 is constant. At a point y on the real line such that m^+ and m^- are defined in a neighborhood of y and $1 - a_{-1}^2m^+(y)m^-(y) = 0$, α_1 has a jump discontinuity. The jump may be determined by the following technique due to Chihara [5].* Choose an interval $[x_0,x]$ containing y, containing no other zero of $1 - a_{-1}^2m^+(z)m^-(z)$, and such that m^+ and m^- are defined on $[x_0,x]$. Let v be a positive number and consider the rectangular contour with vertices $x + iv$, $x_0 + iv$, $x_0 - iv$, and $x - iv$. By the residue theorem

$$\int_{x}^{x_0} M_1(u+iv)du$$

$$+ i\int_{v}^{-v} M_1(x_0+iw)dw$$

$$+ \int_{x_0}^{x} M_1(u-iv)du$$

$$+ i\int_{-v}^{v} M_1(x+iw)dw = 2\pi i \operatorname*{Res}_{z=y} M_1(z).$$

But it is easily seen that $M_1(\bar{z}) = \overline{M_1(z)}$ since both m^+ and m^- have this property, and hence the contour integral may be written as

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*Chihara's result is incorrect by a factor of -1 since the limits of integration used in his argument are reversed.

$$\int_{x_0}^{x} [-M_1(u+iv) + \overline{M_1(u+iv)}]du$$

$$+ i \int_0^v [-\overline{M_1(x_0+iw)} - M_1(x_0+iw) + \overline{M_1(x+iw)} + M_1(x+iw)]dw$$

$$= 2\pi i \operatorname*{Res}_{z=y} M_1(z)$$

or as

$$- 2i \int_{x_0}^{x} \operatorname{Im}[M_1(u+iv)]du$$

$$+ 2i \int_0^v \operatorname{Re}[M_1(x+iw) - M_1(x_0+iw)]dw = 2\pi i \operatorname*{Res}_{z=y} M_1(z).$$

Taking the limit as $v \to 0$ and observing that the second integral approaches zero (since the integrand is continuous) yields

$$\lim_{v \to 0} \frac{1}{\pi} \int_{x_0}^{x} \operatorname{Im}[M_1(u+iv)]du = -\operatorname*{Res}_{z=y} M_1(z).$$

Thus the jump in α_1 at y is $-\operatorname*{Res}_{z=y} M_1(z)$.

In applying the Stieltjes inversion formula on an interval on which m^+ or m^- is not defined, standard theorems on the interchange of limit processes are adequate in many cases.

APPENDIX D

VERIFICATION OF THE ORTHOGONALITY PROPERTY OF α_1, α_2, AND α_4

In this appendix an outline is given of a procedure to verify directly that the polynomials $\{P_n(x)\}_{-\infty}^{\infty}$ and $\{Q_n(x)\}_{-\infty}^{\infty}$ and integrators $\alpha_1(x)$, $\alpha_2(x)$, and $\alpha_4(x)$ introduced in Chapter V satisfy the relation

$$\int_0^{\infty} P_n(x)P_m(x)d\alpha_1(x)$$

$$+ \int_0^{\infty} [P_n(x)Q_m(x)+P_m(x)Q_n(x)]d\alpha_2(x)$$

$$+ \int_0^{\infty} Q_n(x)Q_m(x)d\alpha_4(x) = \delta_{m,n}.$$

Such a procedure serves as protection against any possible errors which may have been made in the preceding steps.

Because each of the integrators is absolutely continuous on the interval $[0, \frac{4k}{m}]$ and is constant outside this interval except for a jump at $\frac{k}{m}(2+\sqrt{5})$, the above relation reduces to

$$\int_0^{4k/m} P_n(x)P_m(x)d\alpha_1(x)$$

$$+ \int_0^{4k/m} [P_n(x)Q_m(x)+P_m(x)Q_n(x)]d\alpha_2(x)$$

$$+ \int_0^{4k/m} Q_n(x)Q_m(x)d\alpha_4(x)$$

$$+ J_{m,n} = \delta_{m,n}$$

where

$$J_{m,n} = \frac{1}{\sqrt{5}} P_n\left(\frac{k}{m}(2+\sqrt{5})\right) P_m\left(\frac{k}{m}(2+\sqrt{5})\right)$$

$$+ \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left[P_n\left(\frac{k}{m}(2+\sqrt{5})\right)Q_m\left(\frac{k}{m}(2+\sqrt{5})\right) + P_m\left(\frac{k}{m}(2+\sqrt{5})\right)Q_n\left(\frac{k}{m}(2+\sqrt{5})\right)\right]$$

$$+ \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 Q_n\left(\frac{k}{m}(2+\sqrt{5})\right)Q_m\left(\frac{k}{m}(2+\sqrt{5})\right).$$

The values of the polynomials at $\frac{k}{m}(2+\sqrt{5})$ may be computed by treating the recurrence relations satisfied by the polynomials as difference equations. The results are

$$P_n\left(\frac{k}{m}(2+\sqrt{5})\right) = \begin{cases} (-1)^n\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}, & n \geq 0, \\ (-1)^n\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}, & n < 0, \end{cases}$$

$$Q_n\left(\frac{k}{m}(2+\sqrt{5})\right) = (-1)^n\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n = \ldots,-1,0,1,\ldots.$$

Therefore

$$J_{m,n} = \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+m}.$$

The integrals may be evaluated by using the substitution $x = \frac{2k}{m}(1+\cos\theta)$, since it was shown in Chapter V that

$$P_n\big(x(\theta)\big) = \begin{cases} \dfrac{(-1)^n\cos(n+\frac{1}{2})\theta}{\cos(\theta/2)}, & n\geq 0, \\[2ex] \dfrac{(-1)^n\sin(n+1)\theta}{\sin\theta}, & n<0, \end{cases}$$

$$Q_n\big(x(\theta)\big) = \frac{(-1)^n\sin n\theta}{\sin\theta}, \qquad n = \ldots,-1,0,1,\ldots.$$

Because the P_n's have different representations for $n\geq 0$ and for $n<0$, the evaluation of the integrals must be broken up into several cases. In each case, however, it can be shown that the evaluation reduces to computing some of the following four integrals:

$$I_1(p,q) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin p\theta\sin q\theta}{1+4\sin^2\theta}\,d\theta,$$

$$I_2(p,q) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin p\theta\sin q\theta\cos\theta}{1+4\sin^2\theta}\,d\theta,$$

$$I_3(p,q) = \frac{1}{\pi}\int_0^{\pi} \frac{\cos p\theta\cos q\theta}{1+4\sin^2\theta}\,d\theta,$$

and

$$I_4(p,q) = \frac{1}{\pi}\int_0^{\pi} \frac{\cos p\theta\cos q\theta\cos\theta}{1+4\sin^2\theta}\,d\theta,$$

where p and q can take on all integral values. Moreover, simple trigonometric identities show that

$$I_1(p,q) = \tfrac{1}{2}[I_3(p-q,0)-I_3(p+q,0)],$$

$$I_2(p,q) = \tfrac{1}{2}[I_1(p+1,q)+I_1(p-1,q)],$$

and

$$I_4(p,q) = \tfrac{1}{2}[I_3(p+1,q)+I_3(p-1,q)];$$

so it is sufficient to compute $I_3(p,q)$. Also since $I_3(-p,q) = I_3(p,-q) = I_3(-p,-q) = I_3(p,q)$, it is sufficient to compute $I_3(p,q)$ for nonnegative values of p and q. Since the integrand is an even function of θ,

$$I_3(p,q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos p\theta \cos q\theta}{1 + 4\sin^2\theta}\, d\theta.$$

But from this representation it follows that $I_3(p,q) = 0$ if $p + q$ is odd, because

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos p\theta \cos q\theta}{1 + 4\sin^2\theta}\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\cos p\theta \cos q\theta}{1 + 4\sin^2\theta}\, d\theta$$

$$+ \frac{1}{2n}\int_{0}^{\pi} \frac{\cos p\theta \cos q\theta}{1 + 4\sin^2\theta}\, d\theta,$$

and making the change of variable $\phi = \theta + \pi$ in the first integral on the right side of this equation yields

$$I_3(p,q) = \tfrac{1}{2}[(-1)^{p+q}I_3(p,q)+I_3(p,q)].$$

To evaluate $I_3(p,q)$ when $p + q$ is even, make the change of variable $z = e^{i\theta}$ so that

$$I_3(p,q) = \frac{1}{2\pi i}\oint_{|z|=1} -\frac{z}{4}\left[\frac{(z^p+z^{-p})(z^q+z^{-q})}{z^4 - 3z^2 + 1}\right]dz.$$

The contour integral can be evaluated by residue theory. The only singularities of the integrand in the interior of the unit circle are simple poles at $\frac{1-\sqrt{5}}{2}$ and $-\frac{1-\sqrt{5}}{2}$ and a multiple pole at the origin. To evaluate the residue at the multiple pole, it is convenient to use the expansion

$$\frac{1}{z^4 - 3z^2 + 1} = \sum_{k=0}^{\infty} \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{2k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{2k+2}\right]z^{2k},$$

which may be obtained by decomposing $\frac{1}{z^4 - 3z^2 + 1}$ by partial fractions and expanding each fraction in a power series. The results for the four integrals for nonnegative values of p and q are

$$I_1(p,q) = \begin{cases} \frac{1}{2\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{|p-q|} - \left(\frac{1-\sqrt{5}}{2}\right)^{p+q}\right], & p+q \text{ even,} \\ 0, & p+q \text{ odd,} \end{cases}$$

$$I_2(p,q) = \begin{cases} 0 & p+q \text{ even,} \\ \frac{1}{4}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{p+q} - \left(\frac{1-\sqrt{5}}{2}\right)^{|p-q|}\right], & p+q \text{ odd,} \end{cases}$$

$$I_3(p,q) = \begin{cases} \frac{1}{2\sqrt{5}}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{|p-q|} + \left(\frac{1-\sqrt{5}}{2}\right)^{p+q}\right], & p+q \text{ even,} \\ 0, & p+q \text{ odd,} \end{cases}$$

$$I_4(p,q) = \begin{cases} 0, & p+q \text{ even,} \\ -\frac{1}{4}\left[\left(\frac{1-\sqrt{5}}{2}\right)^{|p-q|} + \left(\frac{1-\sqrt{5}}{2}\right)^{p+q}\right], & p+q \text{ odd.} \end{cases}$$

For illustrative purposes the verification of the orthogonality property of α_1, α_2, and α_4 in the case where $n \geq 0$, $m \geq 0$ is sketched below. The verification for the other cases is similar.

When $n \geq 0$ and $m \geq 0$,

$$\int_0^{4k/m} P_n(x)P_m(x)d\alpha_1(x)$$

$$+ \int_0^{4k/m} [P_n(x)Q_m(x)+P_m(x)Q_n(x)]d\alpha_2(x)$$

$$+ \int_0^{4k/m} Q_n(x)Q_m(x)d\alpha_4(x) = (-1)^{n+m}\Bigg[\frac{4}{\pi}\int_0^{\pi}\left(\frac{\cos(n+\frac{1}{2})\theta\cos(m+\frac{1}{2})\theta}{\cos^2\frac{\theta}{2}}\right)\left(\frac{\sin^2\theta}{1+4\sin^2\theta}\right)d\theta$$

$$+ \frac{2}{\pi}\int_0^{\pi}\left(\frac{\cos(n+\frac{1}{2})\theta\sin m\theta}{\cos\left(\frac{\theta}{2}\right)\sin\theta}\right)\frac{\sin^2\theta(1-2\cos\theta)}{1+4\sin^2\theta}d\theta$$

$$+ \frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin n\theta\cos(m+\frac{1}{2})\theta}{\cos\frac{\theta}{2}\sin\theta}\right)\frac{\sin^2\theta(1-2\cos\theta)}{1+4\sin^2\theta}d\theta$$

$$+ \frac{2}{\pi}\int_0^{\pi}\frac{\sin n\theta\sin m\theta\sin^2\theta(3-2\cos\theta)}{\sin^2\theta(1+4\sin^2\theta)}d\theta\Bigg]$$

$$= (-1)^{n+m}\Bigg[\frac{4}{\pi}\int_0^{\pi}\left(\frac{\cos(n-m)\theta+\cos(n+m+1)\theta}{1+4\sin^2\theta}\right)(1-\cos\theta)d\theta$$

$$+ \frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin(n+1)\theta\sin m\theta+\sin(m+1)\theta\sin n\theta-2\sin n\theta\sin m\theta}{1+4\sin^2\theta}\right)(1-2\cos\theta)d\theta$$

$$+ \frac{2}{\pi}\int_0^{\pi}\frac{\sin n\theta\sin m\theta}{1+4\sin^2\theta}(3-2\cos\theta)d\theta\Bigg]$$

$$= (-1)^{n+m}\{4[I_3(n-m,0) + I_3(n+m+1,0)] - 4[I_4(n-m,0)$$

$$+ I_4(n+m+1,0)] + 2[I_1(n+1,m) + I_1(n,m+1)$$

$$- 2I_1(n,m)] - 4[I_2(n+1,m) + I_2(n,m+1)$$

$$- 2I_2(n,m)] + 6I_1(n,m) - 4I_2(n,m)\}.$$

Using the known values of I_1, I_2, I_3, and I_4 shows that the sum of the integrals is

$$\begin{cases} 1 - \dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^{n+m}, & n=m \\ -\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^{n+m}, & n\neq m. \end{cases}$$

But $J_{m,n} = (1/\sqrt{5})(1-\sqrt{5}/2)^{n+m}$, and so the orthogonality property is verified in this case.

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